Chapter 5

FINITE-VOLUME METHODS

In Chapter 3, we saw how to derive finite-difference approximations to arbitrary derivatives. In Chapter 4, we saw that the application of a finite-difference approximation to the spatial derivatives in our model PDE's produces a coupled set of ODE's. In this Chapter, we will show how similar semi-discrete forms can be derived using finite-volume approximations in space. Finite-volume methods have become popular in CFD as a result, primarily, of two advantages. First, they ensure that the discretization is conservative, i.e., mass, momentum, and energy are conserved in a discrete sense. While this property can usually be obtained using a finite-difference formulation, it is obtained naturally from a finite-volume formulation. Second, finite-volume methods do not require a coordinate transformation in order to be applied on irregular meshes. As a result they can be applied on *unstructured* meshes consisting of arbitrary polyhedra in three dimensions or arbitrary polygons in two dimensions. This increased flexibility can be used to great advantage in generating grids about arbitrary geometries.

Finite-volume methods are applied to the integral form of the governing equations, either in the form of Eq. 2.1 or Eq. 2.2. Consistent with our emphasis on semi-discrete methods, we will study the latter form, which is

$$\frac{d}{dt}\int_{V(t)} Q dV + \oint_{S(t)} \mathbf{n}.\mathbf{F} dS = \int_{V(t)} P dV \tag{5.1}$$

We will begin by presenting the basic concepts which apply to finite-volume strategies. Next we will give our model equations in the form of Eq. 5.1. This will be followed by several examples which hopefully make these concepts clear.

5.1 Basic Concepts

The basic idea of a finite-volume method is to satisfy the integral form of the conservation law to some degree of approximation for each of many contiguous control volumes which cover the domain of interest. Thus the volume V in Eq. 5.1 is that of a control volume whose shape is dependent on the nature of the grid. In our examples, we will consider only control volumes which do not vary with time. Examining Eq. 5.1, we see that several approximations must be made. The flux is required at the boundary of the control volume, which is a closed surface in three dimensions and a closed contour in two dimensions. This flux must then be integrated to find the net flux through the boundary. Similarly, the source term P must be integrated over the control volume. Next a time-marching method[1] can be applied to find the value of

$$\int_V QdV \tag{5.2}$$

at the next time step. Finally, based on these updated integral values, one can find values of Q at the nodes of the grid, if necessary.

Let us consider these approximations in more detail. First, we note that the average value of Q in a cell with volume V is

$$\bar{Q} \equiv \frac{1}{V}\int_V QdV \tag{5.3}$$

and Eq. 5.1 can be written as

$$V\frac{d}{dt}(\bar{Q}) + \oint_S \mathbf{n}.\mathbf{F}dS = \int_V PdV \tag{5.4}$$

for a control volume which does not vary with time. Thus after applying the time-marching method, we have updated values of the cell-averaged quantities $\bar{Q}$. In order to evaluate the fluxes, which are a function of Q, at the control-volume boundary, Q can be represented within the cell by some piecewise approximation which produces the correct value of $\bar{Q}$. This is a form of interpolation aften referred to as *reconstruction*. As we shall see in our examples, each cell will have a different piecewise approximation to Q. When these are used to calculate $\mathbf{F}(Q)$, they will generally produce different approximations to the flux at the volume boundary of the control volume, that is, the flux will be discontinuous. A nondissipative scheme analogous to centered differencing is obtained by taking the average of these two fluxes. Another approach known as flux-difference splitting is described in Chapter 11.

The basic elements of a finite-volume method are thus the following:

[1]Time-marching methods will be discussed in the next chapter.

1. Given either $\bar{Q}$ or the values of Q at the grid node within each control volume, construct an approximation to $Q(x, y, z)$ in each control volume. Using this approximation, find Q at the control-volume boundary. Evaluate $\mathbf{F}(Q)$ at the boundary, leading in general to different values on either side of the boundary.

2. Apply some strategy for resolving the discontinuity in the fluxes at the control-volume boundary.

3. Integrate the flux to find the net flux through the control-volume boundary using some sort of quadrature.

4. Advance the solution in time to obtain new values of $\bar{Q}$.

The order of accuracy of the method is dependent on each of the approximations. These ideas should be clarified by the examples in the remainder of this chapter.

In order to include diffusive fluxes, the following relation between ∇Q and Q is sometimes used:

$$\int_V \nabla Q dV = \oint_S \mathbf{n} Q dS \tag{5.5}$$

or, in two dimensions,

$$\int_A \nabla Q dA = \oint_C \mathbf{n} Q dl \tag{5.6}$$

where the unit vector $\mathbf{n}$ points outward from the surface or contour.

5.2 Model Equations in Integral Form

5.2.1 The Linear Convection Equation

A two-dimensional form of the linear convection equation can be written as

$$\frac{\partial u}{\partial t} + a \cos\theta \frac{\partial u}{\partial x} + a \sin\theta \frac{\partial u}{\partial y} = 0 \tag{5.7}$$

This PDE governs a simple plane wave convecting the scalar quantity, $u(x, y, t)$ with speed a along a straight line making an angle θ with respect to the x-axis. The one-dimensional form is recovered with $\theta = 0$.

In order to write the two-dimensional linear convection equation in integral form, we first note that for unit speed a it is obtained from the general divergence form, Eq. 2.3, with

$$Q = u \tag{5.8}$$

$$\mathbf{F} = \mathbf{i} u \cos\theta + \mathbf{j} u \sin\theta \tag{5.9}$$

$$P = 0 \tag{5.10}$$

Since Q is a scalar, $\mathbf{F}$ is simply a vector. Substituting these into a two-dimensional form of Eq. 2.2 gives the following integral form

$$\frac{d}{dt}\int_A u dA + \oint_C \mathbf{n}.(\mathbf{i}u\cos\theta + \mathbf{j}u\sin\theta)ds = 0 \tag{5.11}$$

where A is the area of the cell which is bounded by the closed contour C.

5.2.2 The Diffusion Equation

The integral form of the two-dimensional diffusion equation with no source term and a unit diffusion coefficient ν is obtained from the general divergence form, Eq. 2.3, with

$$Q = u \tag{5.12}$$

$$\mathbf{F} = -\nabla u \tag{5.13}$$

$$= -\left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}\right) \tag{5.14}$$

$$P = 0 \tag{5.15}$$

Using these, we find

$$\frac{d}{dt}\int_A u dA = \oint_C \mathbf{n}.\left(\mathbf{i}\frac{\partial u}{\partial x} + \mathbf{j}\frac{\partial u}{\partial y}\right) ds \tag{5.16}$$

to be the integral form of the two-dimensional diffusion equation.

5.3 One-Dimensional Examples

We restrict our attention to a scalar dependent variable u and a scalar flux f, as in the model equations. We consider an equispaced grid with spacing Δx. The nodes of the grid are located at $x_j = j\Delta x$ as usual. Control volume j extends from $x_j - \Delta x/2$ to $x_j + \Delta x/2$, as shown in Fig. ???. We will use the following notation:

$$x_{j-1/2} = x_j - \Delta x/2, \qquad x_{j+1/2} = x_j + \Delta x/2 \tag{5.17}$$

$$u_{j\pm 1/2} = u(x_{j\pm 1/2}), \qquad f_{j\pm 1/2} = f(u_{j\pm 1/2}) \tag{5.18}$$

With these definitions, the cell-average value becomes

$$\bar{u}_j \equiv \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x)dx \tag{5.19}$$

and the integral form becomes

$$\frac{d}{dt}(\Delta x \bar{Q}_j) + f_{j+1/2} - f_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} P dx \tag{5.20}$$

Now with $\xi = x - x_j$, we can expand $u(x)$ in Eq. 5.19 in a Taylor series about x_j to get

$$\begin{aligned}
\bar{u}_j &\equiv \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\left[u_j + \xi\left(\frac{du}{dx}\right)_j + \frac{\xi^2}{2}\left(\frac{d^2u}{dx^2}\right)_j + \frac{\xi^3}{6}\left(\frac{d^3u}{dx^3}\right)_j + \ldots\right] d\xi \\
&= u_j + \frac{\Delta x^2}{24}\left(\frac{d^2u}{dx^2}\right)_j + O(\Delta x^4)
\end{aligned} \tag{5.21}$$

or

$$\bar{u}_j = u_j + O(\Delta x^2) \tag{5.22}$$

where u_j is the value at the center of the cell. Hence the cell-average value and the value at the center of the cell differ by a term of second order.

5.3.1 A Second-Order Approximation to the Convection Equation

In one dimension, the integral form of the linear convection equation, Eq. 5.11, becomes

$$\Delta x \frac{d\bar{u}_j}{dt} + f_{j+1/2} - f_{j-1/2} = 0 \tag{5.23}$$

with $f = u$. We choose a piecewise constant approximation to $u(x)$ in each cell such that

$$u(x) = \bar{u}_j \qquad x_{j-1/2} \le x \le x_{j+1/2} \tag{5.24}$$

Evaluating this at $j + 1/2$ gives

$$f^L_{j+1/2} = f(u^L_{j+1/2}) = u^L_{j+1/2} = \bar{u}_j \tag{5.25}$$

where the L indicates that this approximation to $f_{j+1/2}$ is obtained from the approximation to $u(x)$ in the cell to the *left* of $x_{j+1/2}$, as shown in Fig. ???. The cell to the *right* of $x_{j+1/2}$, which is cell $j + 1$, gives

$$f^R_{j+1/2} = \bar{u}_{j+1} \tag{5.26}$$

Similarly, cell j is the cell to the right of $x_{j-1/2}$, giving

$$f^R_{j-1/2} = \bar{u}_j \tag{5.27}$$

and cell $j-1$ is the cell to the left of $x_{j-1/2}$, giving

$$f^L_{j-1/2} = \bar{u}_{j-1} \tag{5.28}$$

We have now accomplished the first step from the list above, we have defined the fluxes at the cell boundaries in terms of the cell-average data. In this example, the discontinuity in the flux at the cell boundary is resolved by taking the average of the fluxes on either side of the boundary. Thus

$$f_{j+1/2} = \frac{1}{2}(f^L_{j+1/2} + f^R_{j+1/2}) = \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}) \tag{5.29}$$

and

$$f_{j-1/2} = \frac{1}{2}(f^L_{j-1/2} + f^R_{j-1/2}) = \frac{1}{2}(\bar{u}_{j-1} + \bar{u}_j) \tag{5.30}$$

Substituting Eqs. 5.29 and 5.30 into the integral form, Eq. 5.23, we obtain

$$\Delta x \frac{d\bar{u}_j}{dt} + \frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}) - \frac{1}{2}(\bar{u}_{j-1} + \bar{u}_j) = \Delta x \frac{d\bar{u}_j}{dt} + \frac{1}{2}(\bar{u}_{j+1} - \bar{u}_{j-1}) = 0 \tag{5.31}$$

With periodic boundary conditions, this point operator produces the following semi-discrete form:

$$\frac{d\vec{\bar{u}}}{dt} = -\frac{1}{2\Delta x} B_p(-1, 0, 1)\vec{\bar{u}} \tag{5.32}$$

Hence our analysis and understanding of the eigensystem of the matrix $B_p(-1, 0, 1)$ is relevant to finite-volume methods as well as finite-difference methods. Since the eigenvalues of $B_p(-1, 0, 1)$ are pure imaginary, we can conclude that the use of the average of the fluxes on either side of the cell boundary, as in Eqs. 5.29 and 5.30, leads to a nondissipative finite-volume method.

5.3.2 A Fourth-Order Approximation to the Convection Equation

Let us replace the piecewise constant approximation in Section 5.3.1 with a piecewise quadratic approximation as follows

$$u(\xi) = a\xi^2 + b\xi + c \tag{5.33}$$

where ξ is again equal to $x - x_j$. The three parameters a, b, and c are chosen to satisfy the following constraints:

$$\int_{-3\Delta x/2}^{-\Delta x/2} u(\xi) d\xi = \bar{u}_{j-1}$$

$$\int_{-\Delta x/2}^{\Delta x/2} u(\xi) d\xi = \bar{u}_j \tag{5.34}$$

$$\int_{\Delta x/2}^{3\Delta x/2} u(\xi) d\xi = \bar{u}_{j+1}$$

These constraints lead to

$$a = \frac{\bar{u}_{j+1} - 2\bar{u}_j + \bar{u}_{j-1}}{2\Delta x^2}$$

$$b = \frac{\bar{u}_{j+1} - \bar{u}_{j-1}}{2\Delta x} \tag{5.35}$$

$$c = \frac{-\bar{u}_{j-1} + 26\bar{u}_j - \bar{u}_{j+1}}{24}$$

Hence the nodal values which are consistent with this approximation are

$$u_j = u(\xi = 0) = c = \frac{-\bar{u}_{j-1} + 26\bar{u}_j - \bar{u}_{j+1}}{24} \tag{5.36}$$

With these values of a, b, and c, the piecewise quadratic approximation produces the following values at the cell boundaries:

$$u^L_{j+1/2} = \frac{1}{6}(2\bar{u}_{j+1} + 5\bar{u}_j - \bar{u}_{j-1}) \tag{5.37}$$

$$u^R_{j-1/2} = \frac{1}{6}(-\bar{u}_{j+1} + 5\bar{u}_j + 2\bar{u}_{j-1}) \tag{5.38}$$

$$u^R_{j+1/2} = \frac{1}{6}(-\bar{u}_{j+2} + 5\bar{u}_{j+1} + 2\bar{u}_j) \tag{5.39}$$

$$u^L_{j-1/2} = \frac{1}{6}(2\bar{u}_j + 5\bar{u}_{j-1} - \bar{u}_{j-2}) \tag{5.40}$$

using the notation defined in Section 5.3.1. Recalling that $f = u$, we again use the average of the fluxes on either side of the boundary to obtain

$$\begin{aligned} f_{j+1/2} &= \frac{1}{2}[f(u^L_{j+1/2}) + f(u^R_{j+1/2})] \\ &= \frac{1}{12}(-\bar{u}_{j+2} + 7\bar{u}_{j+1} + 7\bar{u}_j - \bar{u}_{j-1}) \end{aligned} \tag{5.41}$$

and

$$\begin{aligned} f_{j-1/2} &= \frac{1}{2}[f(u^L_{j-1/2}) + f(u^R_{j-1/2})] \\ &= \frac{1}{12}(-\bar{u}_{j+1} + 7\bar{u}_j + 7\bar{u}_{j-1} - \bar{u}_{j-2}) \end{aligned} \tag{5.42}$$

Substituting these expressions into the integral form, Eq. 5.23, gives

$$\Delta x \frac{d\bar{u}_j}{dt} + \frac{1}{12}(-\bar{u}_{j+2} + 8\bar{u}_{j+1} - 8\bar{u}_{j-1} + \bar{u}_{j-2}) = 0 \tag{5.43}$$

With periodic boundary conditions, the following semi-discrete form is obtained:

$$\frac{d\vec{\bar{u}}}{dt} = -\frac{1}{12\Delta x} B_p(1, -8, 0, 8, -1)\vec{\bar{u}} \tag{5.44}$$

This is a system of ODE's governing the evolution of the cell-average data.

From Eq. 5.36, we have the following relation between the nodal data and the cell-average data

$$\vec{u} = M\vec{\bar{u}} \quad \text{with} \quad M = \frac{1}{24} B_p(-1, 26, -1) \tag{5.45}$$

Premultiplying Eq. 5.44 by M and inserting the product $M^{-1}M$, we get

$$\frac{dM\vec{\bar{u}}}{dt} = -\frac{1}{12\Delta x} M B_p(1, -8, 0, 8, -1) M^{-1} M\vec{\bar{u}} \tag{5.46}$$

Noting that

$$M B_p(1, -8, 0, 8, -1) M^{-1} = B_p(1, -8, 0, 8, -1) \tag{5.47}$$

we obtain

$$\frac{d\vec{u}}{dt} = -\frac{1}{12\Delta x} B_p(1, -8, 0, 8, -1)\vec{u} \tag{5.48}$$

This is a system of ODE's governing the evolution of the nodal data and is entirely equivalent to a fourth-order finite-difference approximation.

5.3.3 A Second-Order Approximation to the Diffusion Equation

In this section, we describe two approaches to deriving a finite-volume approximation to the diffusion equation. The first approach is simpler to extend to multidimensions, while the second approach is more suited to extension to higher order.

In one dimension, the integral form of the diffusion equation, Eq. 5.16, becomes

$$\Delta x \frac{d\bar{u}_j}{dt} + f_{j+1/2} - f_{j-1/2} = 0 \tag{5.49}$$

with $f = -\nabla u = -\partial u/\partial x$. Also, Eq. 5.6 becomes

$$\int_a^b \frac{\partial u}{\partial x} dx = u(b) - u(a) \tag{5.50}$$

Assuming that the nodal values of u are known at the start of the time step, we can thus write the following expression for the average value of the gradient of u over the interval $x_j \leq x \leq x_{j+1}$:

$$\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} \frac{\partial u}{\partial x} dx = \frac{1}{\Delta x}(u_{j+1} - u_j) \tag{5.51}$$

From Eq. 5.22, we know that the value of a continuous function at the center of a given interval is equal to the average value of the function over the interval to second-order accuracy. Hence, to second-order, we can write

$$f_{j+1/2} = -\left(\frac{\partial u}{\partial x}\right)_{j+1/2} = -\frac{1}{\Delta x}(u_{j+1} - u_j) \tag{5.52}$$

Similarly,

$$f_{j-1/2} = -\frac{1}{\Delta x}(u_j - u_{j-1}) \tag{5.53}$$

Substituting these into the integral form, Eq. 5.49, we obtain

$$\Delta x \frac{d\bar{u}_j}{dt} = \frac{1}{\Delta x}(u_{j-1} - 2u_j + u_{j+1}) \tag{5.54}$$

Finally, from Eq. 5.22, we can replace u_j with $\bar{u}_j$ to second order, giving

$$\frac{d\bar{u}_j}{dt} = \frac{1}{\Delta x^2}(\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1}) \tag{5.55}$$

or, with Dirichlet boundary conditions,

$$\frac{d\vec{\bar{u}}}{dt} = \frac{1}{\Delta x^2} B(1, -2, 1)\vec{\bar{u}} + \left(\vec{bc}\right) \tag{5.56}$$

This provides a semi-discrete finite-volume approximation to the diffusion equation and we see that the properties of the matrix $B(1, -2, 1)$ are relevant to the study of finite-volume methods.

For our second approach, we use a piecewise quadratic approximation as in Section 5.3.2. From Eq. 5.33 we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} = 2a\xi + b \tag{5.57}$$

with a and b given in Eq. 5.35. With $f = -\partial u/\partial x$, this gives

$$f^R_{j+1/2} = f^L_{j+1/2} = -\frac{1}{\Delta x}(\bar{u}_{j+1} - \bar{u}_j) \tag{5.58}$$

$$f^R_{j-1/2} = f^L_{j-1/2} = -\frac{1}{\Delta x}(\bar{u}_j - \bar{u}_{j-1}) \tag{5.59}$$

Notice that there is no discontinuity in the flux at the cell boundary. This produces

$$\frac{d\bar{u}_j}{dt} = \frac{1}{\Delta x^2}(\bar{u}_{j-1} - 2\bar{u}_j + \bar{u}_{j+1}) \tag{5.60}$$

which is identical to Eq. 5.55. The resulting semi-discrete form with periodic boundary conditions is

$$\frac{d\vec{\bar{u}}}{dt} = \frac{1}{\Delta x^2} B_p(1, -2, 1)\vec{\bar{u}} \tag{5.61}$$

which is written entirely in terms of cell-average data. An expression for the evolution of the nodal data can be found using the procedure following Eq. 5.45, leading to

$$\frac{d\vec{u}}{dt} = \frac{1}{\Delta x^2} B_p(1, -2, 1)\vec{u} \tag{5.62}$$

This is identical to the expression obtained using second-order centered differences.

5.4 A Two-Dimensional Example

The above one-dimensional examples of finite-volume approximations obscure some of the practical aspects of such methods. Thus our final example is a finite-volume approximation to the two-dimensional linear convection equation on a grid consisting of regular triangles, as shown in Figure ???. As in Section 5.3.1, we use a piecewise constant approximation in each control volume and the flux at the control volume boundary is the average of the fluxes obtained on either side of the boundary. The nodal data are stored at the vertices of the triangles formed by the grid. The control volumes are regular hexagons with area A, as shown in the figure. Δ is the length of the sides of the triangles, and ℓ is the length of the sides of the hexagons. The following relations hold between ℓ, Δ, and A.

$$\begin{aligned} \ell &= \frac{1}{\sqrt{3}}\Delta \\ A &= \frac{3\sqrt{3}}{2}\ell^2 \\ \frac{\ell}{A} &= \frac{2}{3\Delta} \end{aligned} \tag{5.63}$$

The two-dimensional form of the conservation law is

$$\frac{d}{dt}\int_A Q dV + \oint_C \mathbf{n}.\mathbf{F} dl = 0 \tag{5.64}$$

where we have ignored the source term. The contour in the line integral is composed of the sides of the hexagon. Since these sides are all straight, the unit normals can be taken outside the integral and the flux balance is given by

$$\frac{d}{dt}\int_A Q\, dA + \sum_{\nu=0}^{5} \mathbf{n}_\nu \cdot \int_\nu \mathbf{F} dl = 0$$

where ν indexes a side of the hexagon as shown in the figure. A list of the normals for the mesh orientation shown is given in Table 5.1.

Side, ν	Outward Normal, $\mathbf{n}$
0	$(\mathbf{i} - \sqrt{3}\,\mathbf{j})/2$
1	$\mathbf{i}$
2	$(\mathbf{i} + \sqrt{3}\,\mathbf{j})/2$
3	$(-\mathbf{i} + \sqrt{3}\,\mathbf{j})/2$
4	$-\mathbf{i}$
5	$(-\mathbf{i} - \sqrt{3}\,\mathbf{j})/2$

Table 5.1. Outward normals, see Fig 5.2.
$\mathbf{i}$ and $\mathbf{j}$ are unit normals along x and y, respectively.

For Eq. 5.11, the two-dimensional linear convection equation, we have for side ν

$$\mathbf{n}_\nu \cdot \int_\nu \mathbf{F} ds = \mathbf{n}_\nu \cdot (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) \int_{-\ell/2}^{\ell/2} u_\nu(\xi) d\xi \tag{5.65}$$

where ξ is a length measured from the middle of a side ν. Making the change of variable $z = \xi/\ell$, one has the expression

$$\int_{-\ell/2}^{\ell/2} u(\xi) d\xi = \ell \int_{-1/2}^{1/2} u(z) dz \tag{5.66}$$

Then, in terms of u and the hexagon area A, we have

$$\frac{d}{dt}\int_A u \, dA + \sum_{\nu=0}^{5} \mathbf{n}_\nu \cdot (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta) \left[\ell \int_{-1/2}^{1/2} u(z) dz \right]_\nu = 0 \tag{5.67}$$

The values of $\mathbf{n}_\nu \cdot (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$ are given by the expressions in Table 5.2. There are no numerical approximations in Eq. 5.67. That is, if the integrals in the equation are evaluated exactly, the integrated time rate of change of the integral of u over the area of the hexagon is known exactly.

Side, ν	$\mathbf{n}_\nu \cdot (\mathbf{i}\cos\theta + \mathbf{j}\sin\theta)$
0	$(\cos\theta - \sqrt{3}\sin\theta)/2$
1	$\cos\theta$
2	$(\cos\theta + \sqrt{3}\sin\theta)/2$
3	$(-\cos\theta + \sqrt{3}\sin\theta)/2$
4	$-\cos\theta$
5	$(-\cos\theta - \sqrt{3}\sin\theta)/2$

Table 5.2. Weights of flux integrals, see Eq. 5.67.

Introducing the piecewise-constant approximation $u = u_p$ over the entire hexagon, we have

$$\int_A u\, dA = A u_p \tag{5.68}$$

The approximation to the flux integral is also trivial in the piecewise-constant case. Taking the average of the flux on either side of each side of the hexagon gives

$$\int_1 u(z)dz = \frac{u_p + u_a}{2} \int_{-1/2}^{1/2} dz = \frac{u_p + u_a}{2} \tag{5.69}$$

Similarly, we have for the other five sides:

$$\int_2 u(z)dz = \frac{u_p + u_b}{2} \tag{5.70}$$

$$\int_3 u(z)dz = \frac{u_p + u_c}{2} \tag{5.71}$$

$$\int_4 u(z)dz = \frac{u_p + u_d}{2} \tag{5.72}$$

$$\int_5 u(z)dz = \frac{u_p + u_e}{2} \tag{5.73}$$

$$\int_0 u(z)dz = \frac{u_p + u_f}{2} \tag{5.74}$$

Substituting these into Eq. 5.67, along with the expressions in Table 5.2, we obtain

$$A\frac{du_p}{dt} + \frac{\ell}{2}[(2\cos\theta)(u_a - u_d) + (\cos\theta + \sqrt{3}\sin\theta)(u_b - u_e) \\ +(-\cos\theta + \sqrt{3}\sin\theta)(u_c - u_f)] = 0 \tag{5.75}$$

or

$$\frac{du}{dt} + \frac{1}{2\Delta}[(2\cos\theta)(u_a - u_d) + (\cos\theta + \sqrt{3}\sin\theta)(u_b - u_e) \\ +(-\cos\theta + \sqrt{3}\sin\theta)(u_c - u_f)] = 0 \tag{5.76}$$

Taylor series expansions of the terms in this finite-volume operator show that it is of second-order accuracy, as we might expect.